Densities for nonlinear Gaussian functionals: from existence to some striking applications

Marta Sanz-Solé, University of Barcelona
http://www.ub.edu/plie/Sanz-Sole

Ateneo IMUva
Valladolid, 27.10.2016

Plan of the lecture

1. An introduction to Malliavin calculus
   1.1 Densities and “integration by parts” formulas
   1.2 Densities of diffusion processes and partial differential equations
   1.3 Malliavin calculus in a nutshell
2. Applications
   2.1 Hitting probabilities. General criteria
   2.2 Hitting probabilities for SPDEs

Stochastic Calculus of Variations *(Malliavin Calculus) consists, in brief, in constructing and exploiting natural differentiable structures on abstract probability spaces; in other words, Stochastic Calculus of Variations proceeds from a merging of differential calculus and probability theory.*

Paul Malliavin and Anton Thalmaier

Stochastic Calculus of Variations in Mathematical Finance, 2006

Motivation

- To give a probabilistic proof of Hörmander's theorem on hypoellipticity of differential operators in square form. (Existence of smooth densities).
- To understand better the interplay between Kolmogorov's equation for densities of diffusions and stochastic differential equations.

The very basic notions in probability theory

- Probability space $(\Omega, \mathcal{G}, P)$ (frame for random experiences).
- Random vectors $X : (\Omega, \mathcal{G}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ (shift to a numerical model).
- Probability law of X: $P \circ X^{-1}$ (probability on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$).

If $P \circ X^{-1}$ is absolutely continuous:

$$P\{\omega \in \Omega : X(\omega) \in A\} = \int_A f(x) dx.$$

Having the expression and properties of f is crucial in many computations on probabilistic models and in statistical analysis.

Densities and Integration by Parts

Notation

$F : \Omega \to \mathbb{R}^n$, random vector, $\alpha = (\alpha_1, \ldots, \alpha_r) \in \{1, \ldots, n\}^r$,
$|\alpha| = \sum_{i=1}^r \alpha_i$,
$\varphi : \mathbb{R}^n \to \mathbb{R}$, $\partial_\alpha \varphi = \partial^{|\alpha|}_{\alpha_1, \ldots, \alpha_r} \varphi$.

Definition

F satisfies an integration by parts formula (IBP) of degree $|\alpha|$ if there exists a random variable $H_\alpha(F) \in L^1(\Omega)$ such that

$$\boxed{E((\partial_\alpha \varphi)(F)) = E(\varphi(F) H_\alpha(F)),}$$

for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$.

Notation

$F : \Omega \to \mathbb{R}^n$, random vector, $\alpha = (\alpha_1, \ldots, \alpha_r) \in \{1, \ldots, n\}^r$,
$|\alpha| = \sum_{i=1}^r \alpha_i$,
$\varphi : \mathbb{R}^n \to \mathbb{R}$, $\partial_\alpha \varphi = \partial^{|\alpha|}_{\alpha_1, \ldots, \alpha_r} \varphi$.

Definition

F satisfies an integration by parts formula (IBP) of degree $|\alpha|$ if there exists a random variable $H_\alpha(F) \in L^1(\Omega)$ such that

$$\boxed{E((\partial_\alpha \varphi)(F)) = E(\varphi(F) H_\alpha(F)),}$$

for any $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$.

Theorem

1. *Assume that the IBP formula holds for* $\alpha = (1, \ldots, 1)$. *Then the probability law of* F *has a density* $p(x)$ *with respect to Lebesgue measure on* $\mathbb{R}^n$, *and*

$$\boxed{p(x) = E\big(1_{(x \le F)} H_{(1,\ldots,1)}(F)\big).}$$

In particular, p *is continuous.*

2. *Assume that for any multiindex* α *the IBP holds true. Then* $p \in \mathcal{C}^{|\alpha|}(\mathbb{R}^n)$ *and*

$$\boxed{\partial_\alpha p(x) = (-1)^{|\alpha|} E\big(\mathbf{1}_{(x \le F)} H_{\alpha+1}(F)\big),}$$

where $\alpha + 1 := (\alpha_1 + 1, \ldots, \alpha_d + 1)$.

Non-rigorous argument

$$p(x)dx = P\{F \in dx\}, \text{ Radon-Nikodym Theorem}$$
$$P\{F \in A\} = E(1_A(F)).$$

Thus, $(A = \{dx\} \sim \{x\})$

$$\begin{aligned} p(x) &= E\big(\delta_0(F - x)\big) \\ &= E\big((\partial_{1,\dots,1}\mathbf{1}_{[0,\infty)})(F - x)\big) \\ &= E\big(\mathbf{1}_{[0,\infty)}(F - x)H_{(1,\dots,1)}(F)\big). \end{aligned}$$

Questions

- How to check the integration-by-parts formula for non-Gaussian random variables?
- Is there an explicit expression for $H_\alpha(X)$?

Malliavin Calculus provides answers to these questions for random variables F which are functionals of a Gaussian process $\{Z_w, w \in I\}$:

$$F = \Phi(Z_w, w \in I).$$

Questions

- How to check the integration-by-parts formula for non-Gaussian random variables?
- Is there an explicit expression for $H_\alpha(X)$?

Malliavin Calculus provides answers to these questions for random variables F which are functionals of a Gaussian process $\{Z_w, w \in I\}$:

$$F = \Phi(Z_w, w \in I).$$

A trivial example

$n = 1,\ F \overset{\mathcal{L}}{=} \mathrm{N}(0,1)$:

$$
\begin{aligned}
E((\varphi')(F)) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi'(y) \exp\left(-\frac{y^2}{2}\right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(y) y \exp\left(-\frac{y^2}{2}\right) dy \\
&= E(\varphi(F)F).
\end{aligned}
$$

Thus (IBP) of degree 1 holds with $H_1(F) = F$.

The fundamental Malliavin's lemma (1978)

μ is a finite measure on $\mathbb{R}^n$. Assume that $\forall \varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} \partial_i \varphi d\mu \right| \leq c_i \|\varphi\|_\infty, \ 1 \leq i \leq n. \tag{1}$$

Then μ is a.c., and the density belongs to $L^{\frac{n}{n-1}}(\mathbb{R}^n)$.

Example: μ law of the random vector F. Condition (1) is

$$|E((\partial_i \varphi)(F))| \leq c_i \|\varphi\|_\infty, \ 1 \leq i \leq n.$$

Remark: The IBP formula for $\alpha = (0, \dots, \overset{(i)}{1}, \dots, 0)$ yields

$$|E((\partial_i \varphi)(F))| = |E(\varphi(F) H_i(F))| \leq \|\varphi\|_\infty E(|H_i(F)|) \leq c_i \|\varphi\|_\infty.$$

The fundamental Malliavin's lemma (1978)

μ is a finite measure on $\mathbb{R}^n$. Assume that $\forall \varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} \partial_i \varphi d\mu \right| \leq c_i \|\varphi\|_\infty, \; 1 \leq i \leq n. \tag{1}$$

Then μ is a.c., and the density belongs to $L^{\frac{n}{n-1}}(\mathbb{R}^n)$.

Example: μ law of the random vector F. Condition (1) is

$$|E((\partial_i \varphi)(F))| \leq c_i \|\varphi\|_\infty, \; 1 \leq i \leq n.$$

Remark: The IBP formula for $\alpha = (0, \dots, \overset{(i)}{1}, \dots, 0)$ yields

$$|E((\partial_i \varphi)(F))| = |E(\varphi(F) H_i(F))| \leq \|\varphi\|_\infty E(|H_i(F)|) \leq c_i \|\varphi\|_\infty.$$

The fundamental Malliavin's lemma (1978)

μ is a finite measure on $\mathbb{R}^n$. Assume that $\forall \varphi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} \partial_i \varphi d\mu \right| \leq c_i \|\varphi\|_\infty, \ 1 \leq i \leq n. \tag{1}$$

Then μ is a.c., and the density belongs to $L^{\frac{n}{n-1}}(\mathbb{R}^n)$.

Example: μ law of the random vector F. Condition (1) is

$$|E((\partial_i \varphi)(F))| \leq c_i \|\varphi\|_\infty, \ 1 \leq i \leq n.$$

Remark: The IBP formula for $\alpha = (0, \ldots, \overset{(i)}{1}, \ldots, 0)$ yields

$$|E((\partial_i \varphi)(F))| = |E(\varphi(F) H_i(F))| \leq \|\varphi\|_\infty E(|H_i(F)|) \leq c_i \|\varphi\|_\infty.$$

Stochastic Calculus of Variations (Malliavin Calculus) consists, in brief, in constructing and exploiting natural differentiable structures on abstract probability spaces.

Densities of diffusion processes and SPDEs

The fundamental example: Brownian motion

$\{B_t = (B_t^1, \dots, B_t^n), t \geq 0\}$ Gaussian

- $E(B_t^i) = 0$,
- $E(B_t^i B_s^j) = \delta_i^j \min(s,t)$.

The density of B_t, $t > 0$, is

$$p_t(y) = (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|y|^2}{2t}\right),$$

and satisfies the heat equation

$$\begin{cases} \frac{\partial p_t(y)}{\partial t} & = \frac{1}{2}\Delta p_t(y), \\ p_0(t) & = \delta_0, \end{cases}$$

where $\Delta = \sum_{i=1}^n \partial_i^2$.

Diffusion processes

A diffusion is a continuous process $\{X_t = (X_t^1, \dots, X_t^n), t \geq 0\}$ satisfying

$$E\left[X_{t+h}^i - X_t^i | X_s, 0 \leq s \leq t\right] = b^i(X_t)h + o(h),$$
$$E\left[\left(X_{t+h}^i - X_t^i - b^i(X_t)h\right)\left(X_{t+h}^j - X_t^i - b^j(X_t)h\right)\right] = a^{ij}(X_t)h + o(h),$$

$h > 0$, with smooth real functions b^i, a^{ij}.

Fact: X is a Markov process with transition density function

$$P\{X_{u+t} \in dy | X_s, 0 \leq s \leq u, X_u = x\} = p_t^x(y)dy.$$

(Kolmogorov, 1931, Feller, ...)

Kolmogorov's equations

Notation

$$(\mathcal{L}f)(x) = \frac{1}{2}\sum_{i,j} a^{ij}(x)\partial_i\partial_j f(x) + \sum_i b^i(x)\partial_i f(x),$$

$$(\mathcal{L}^* f)(x) = \frac{1}{2}\sum_{i,j} \partial_i\partial_j[a^{ij}(x)f(x)] - \sum_i \partial_i[b^i(x)f(x)].$$

Forward equation: $\frac{\partial}{\partial t}p_t^x(y) = \mathcal{L}^* p_t^x(y)$, $y \in \mathbb{R}^n$ fixed.

Backward equation: $\frac{\partial}{\partial t}p_t^x(y) = \mathcal{L} p_t^x(y)$, $x \in \mathbb{R}^n$ fixed.

Initial condition: $p_0^x(y) = \delta_x$.

Diffusion processes and Itô's equation (Itô, 1946)

- $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n \otimes \mathbb{R}^d$, $b : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, Lipschitz continuous, linear growth,
- $\{B_t, t \geq 0\}$ d-dimensional Brownian motion.

The stochastic differential equation on $\mathbb{R}^n$

$$\begin{cases} dX_t^x & = \sigma(X_t^x)dB_t + b(X_t^x)dt, \\ X_0^x & = x, \end{cases}$$

has a unique solution.
$\{X_t^x, t \geq 0\}$ is a diffusion process with $a = \sigma\sigma^t$.

Kolmogorov's equation for the law of X_t^x

- Itô formula, $f \in \mathcal{C}_K^\infty((0,\infty) \times \mathbb{R}^n)$

$$f(t, X_t^x) = f(0,x) + \int_0^t \left[\frac{\partial}{\partial s} + \mathcal{L}\right] (f(s, X_s^x))ds + \text{martingale.}$$

- Take expectations

$$E\left(\int_0^\infty \left[\frac{\partial}{\partial s} + \mathcal{L}\right] (f(s, X_s^x))ds\right) = 0. \tag{2}$$

- Introduce a distribution

$$\alpha(g) := E\left(\int_0^\infty g(s, X_s^x)ds\right) = \int_0^\infty \int_{\mathbb{R}^n} g(s,y) P(X_s^x \in dy)ds.$$

Then (2) reads

$$\alpha\left(\frac{\partial}{\partial t}+\mathcal{L}\right)=0 \Leftrightarrow \left(-\frac{\partial}{\partial t}+\mathcal{L}^*\right)\alpha=0.$$

If $\left(-\frac{\partial}{\partial t}+\mathcal{L}^*\right)$ is hypoelliptic, then α is a smooth function on $(0,\infty)\times\mathbb{R}^n$:

$$\alpha(f)=\int_0^\infty\int_{\mathbb{R}^n} f(s,y)p_s^x(y)dsdy.$$

and

$$\frac{\partial p^x}{\partial t}=\mathcal{L}^* p^x.$$

Hörmander's theorem

- $\sigma_i^j, b^i \in \mathcal{C}_b^\infty$.
- Define vector fields

$$A_j = \sigma_j^i \partial_i, \; j = 1, \ldots, n,$$
$$A_0 = b - \frac{1}{2} \sum_{l=1}^{d} A_l^\nabla A_l, \; A_l^\nabla A_k = A_l^j \partial_j A_k^i \partial_i,$$
$$[A_j, A_k] = A_j^\nabla A_k - A_k^\nabla A_j.$$

Theorem *At each point $x \in \mathbb{R}^n$, the vector space spanned by*

$$A_1, \ldots, A_d, \; [A_i, A_j], 0 \le i, j \le d, \; [A_i, [A_j, A_k]], 0 \le i, j, k \le d, \ldots,$$

is $\mathbb{R}^n$. Then $\left(\frac{\partial}{\partial t} - \mathcal{L}^*\right)$ is hypoelliptic.

Malliavin Calculus in a Nutshell

Malliavin's project

Under Hörmander's assumptions, to prove an IBP formula:

$$\boxed{E((\partial_\alpha\varphi)(X_t^x)) = E(\varphi(X_t^x)H_\alpha(X_t^x)),}$$

$t > 0$.

In other words, to give an entirely probabilistic proof of Hörmander's theorem.

Warming up: the Finite Dimensional Case

Let

$$\mu_m(dx) = (2\pi)^{-\frac{m}{2}} \exp\Big(-\frac{|x|^2}{2}\Big)\, dx.$$

On the probability space $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \mu_m)$, consider the (smooth) random vector $F : \mathbb{R}^m \to \mathbb{R}^n$.

Find an IBP formula for F: $E((\partial_i \varphi)(F)) = \cdots$.

By the chain rule,

$$\big\langle \nabla\big(\varphi(F(x))\big), \nabla F^l(x) \big\rangle = \big(A(x)(\nabla^T \varphi)(F(x))\big)_l,$$

$l = 1, \cdots, n$, where $A(x) = \big(\langle \nabla F^i(x), \nabla F^j(x) \rangle\big)_{1 \le i,j \le n}$.

Linear system for $(\partial_i \varphi)(F)$.

If the matrix $A(x)$ is inversible, one gets

$$(\partial_i \varphi)(F(x)) = \sum_{l=1}^{n} \left\langle \nabla\big(\varphi(F(x))\big), A_{il}^{-1}(x) \nabla F^l(x) \right\rangle, i = 1, \dots, n.$$

By taking expectations (integration wrt μ_m)

$$\begin{aligned} E_m\big((\partial_i \varphi)(F)\big) &= \sum_{l=1}^{n} E_m \langle \nabla\big(\varphi(F)\big), A_{il}^{-1} \nabla F^l \rangle \\ &= \sum_{l=1}^{n} E_m\big(\varphi(F) \delta_m (A_{il}^{-1} \nabla F^l)\big), \quad \delta_m \text{ adjoint of } \nabla, \\ &= E_m\big(\varphi(F) H_i(F, 1)\big), \end{aligned}$$

with

$$H_i(F, 1) = \sum_{l=1}^{n} \delta_m (A_{il}^{-1} \nabla F^l).$$

Main ingredients and assumptions

- Gradient operator ∇,
- F sufficiently smooth in terms of ∇,
- *Gradient covariance* matrix $A(x)$,
- $A(x)$ is inversible,
- Adjoint of ∇, δ_m: $E_m\langle\nabla f, \varphi\rangle = E_m(f\delta_m\varphi)$.

Back to Malliavin's description

Stochastic Calculus of Variations (Malliavin Calculus) consists, in brief, in constructing and exploiting natural differentiable structures on **abstract probability spaces***; in other words, Stochastic Calculus of Variations proceeds from a merging of differential calculus and probability theory.*

From finite to infinite dimensions

Infinite dimensional analogue of $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m), \mu_m)$ (Gross, 1965):

Abstract Wiener Space $(\mathcal{W}, \mathcal{G}, H, \mu)$, where

- $\mathcal{W} = \mathcal{C}([0, T]; \mathbb{R}^d)$,
- $\mathcal{G} = \mathcal{B}(\mathcal{C}([0, T]; \mathbb{R}^d))$,
- μ law of Brownian motion,
- $H \subset \mathcal{W}$, $i : H \to \mathcal{W}$ is continuous, $i(H)$ is dense in $\mathcal{W}$ (Cameron-Martin space).

$$H = \left\{ h : [0, T] \to \mathbb{R}^d, h^i(t) = \int_0^t \dot{h}^i(s) ds, \int_0^T |\dot{h}(s)|^2 < \infty \right\}.$$

Main ingredients

- Malliavin derivative D (∇)
- Sobolev type spaces defined using D: $\mathbb{D}^{k,p}$
- Malliavin matrix $\langle DF^i, DF^j \rangle_H$: (A)
- Inversibility of the Malliavin matrix
- Adjoint of D (Skorohod integral δ): $E(\langle Df, u \rangle_H) = E(F\delta(u))$, (δ_m)

... And the associated calculus.

The derivative operator

$B(h) = \int_0^T h(s)dB_s$, $h \in H$, Gaussian r.v.

Smooth functionals

$$F = f\big(B(h_1), \dots, B(h_n)\big),$$

$f \in \mathcal{C}_p^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in H$, $n \geq 1$.

For $F \in \mathcal{S}$, define

$$DF = \sum_{i=1}^{n} \partial_i f\big(B(h_1), \dots, B(h_n)\big) h_i.$$

This is a H-valued random variable.

Extension

D is closable as an operator from $L^p(\Omega)$ to $L^p(\Omega; H)$, for any $p \geq 1$. That is, if

- $\{F_n, n \geq 1\} \subset \mathcal{S}$, $F_n \xrightarrow{L^p(\Omega)} 0$,
- $\{DF_n, n \geq 1\} \xrightarrow{L^p(\Omega;H)} G$,

then $G = 0$.

Let $\mathbb{D}^{1,p}$ be the closure of the set $\mathcal{S}$ with respect to the seminorm

$$||F||_{1,p} = \left(E(|F|^p) + E(||DF||_H^p)\right)^{\frac{1}{p}}.$$

$\mathbb{D}^{1,p}$ is the domain of the operator D in $L^p(\Omega)$.

Other approaches

- Ornstein-Uhlenbeck operator (Malliavin)
- $\langle DF, h \rangle_H$ directional derivative in H (Bismut)
- Wiener chaos decomposition (Meyer)

Contributors

Malliavin, Bismut, Stroock, Ikeda, Watanabe, Bouleau, Hirsch, Meyer, Kusuoka, Shigekawa, Nualart, Bell, Mohammed, Ocone, Zakai, ...

Where is Malliavin Calculus useful?

In the analysis of densities of functionals of Gaussian processes

Examples of such functionals

- $\{X_t^x, t \geq 0\}$, solution to the Itô equation.
- Maximum of Gaussian continuous processes.
- Stochastic differential and partial differential equations.

Problems

- Existence of density and its properties, strict positivity, etc.
- Upper and lower bounds of Gaussian type.
- Analysis of the effect of perturbations of the Gaussian process.
- ...

Applications

- Anticipating stochastic calculus. Stochastic calculus with respect to (rough) Gaussian processes (*Zakai, Nualart, Pardoux, Tindel, fractional Brownian motion community*, ...)
- Mathematical finance: computation of price sensitivities (*Greeks*) (*Kusuoka–Ninomiya, Fournié-Lasry-Lebuchoux-Lions, Kohatsu-Higa, Malliavin and co-authors*,...)
- **Probabilistic potential theory for non-Markovian processes** (*R. Dalang, D. Khosnevishan, M. S.-S, Y. Xiao*, ...)
- ...

Hitting Probabilities

Introduction

$\{v(x), x \in \mathbb{R}^m\}$ is a $\mathbb{R}^d$-valued random field defined on a probability space $(\Omega, \mathcal{G}, P)$

$$v : \Omega \times \mathbb{R}^m \to \mathbb{R}^d.$$

Question

How many sample paths of $v(\omega)$ visit a deterministic set A?

Prove upper and lower bounds for the hitting probabilities

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} := P\{\omega : v(\omega)(Q) \cap A \neq \emptyset\}}$$

in terms of the capacity or the Hausdorff measure of A.

Factors

- Regularity (or roughness) of the sample paths $v(\omega)$.
- Size and geometry of A.
- The dimensions m and d.

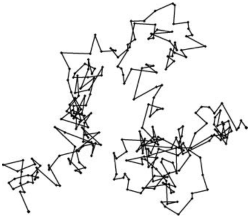


Figure : sample paths of a 2-d Brownian motion

Retated problems

- Characterization of the polar sets A:

$$P\{v(Q) \cap A \neq \emptyset\} = 0.$$

  Maximal solutions of elliptic equations.

- Hausdorff dimension (a.s.) of $v(Q) := \{v(z), z \in Q\}$.
- Study of level sets $\mathcal{L}(u; z) = \{y \in \mathbb{R}^m : u(y) = z\}$,

$$\cdots \leq P\{\mathcal{L}(u; z) \cap E \neq \emptyset\} \leq \cdots$$

  $E \subset \mathbb{R}^m$.

- $\cdots$

Bessel-Riesz capacity

For $\beta \in \mathbb{R}$, $E \in \mathcal{B}(\mathbb{R}^d)$,

$$\boxed{\mathrm{Cap}_\beta(E) = \left[\inf_{\mu \in \mathcal{P}(E)} I_\beta(\mu)\right]^{-1}.}$$

Energy

$$I_\beta(\mu) = \int_E \int_E K_\beta(\|x-y\|)\mu(dx)\mu(dy),$$

μ probability on E.

Bessel-Riesz kernel

$$K_\beta(r) = \begin{cases} r^{-\beta}, & \text{if } \beta > 0, \\ \log_+\left(\frac{1}{r}\right), & \text{if } \beta = 0, \\ 1, & \text{if } \beta < 0. \end{cases}$$

Hausdorff measure

For $\beta \in [0, \infty[$, $E \in \mathcal{B}(\mathbb{R}^d)$:

$$\boxed{\mathcal{H}_\beta(E) = \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_{i=1}^\infty (2r_i)^\beta : E \subset \cup_{i=1}^\infty B_{r_i}(x_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\}.}$$

For $\beta \in]-\infty, 0[$, $E \in \mathcal{B}(\mathbb{R}^d)$, $\mathcal{H}_\beta(E) = \infty$.

A useful fact relating capacities and Hausdorff measures

For $\beta_1 > \beta_2 > 0$ and compact E,

$$\mathrm{Cap}_{\beta_1}(E) > 0 \Longrightarrow \mathcal{H}_{\beta_1}(E) > 0 \Longrightarrow \mathrm{Cap}_{\beta_2}(E) > 0$$

(Frostman's theorem).

Hausdorff measure

For $\beta \in [0, \infty[$, $E \in \mathcal{B}(\mathbb{R}^d)$:

$$\boxed{\mathcal{H}_\beta(E) = \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : E \subset \cup_{i=1}^{\infty} B_{r_i}(x_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\}.}$$

For $\beta \in]-\infty, 0[$, $E \in \mathcal{B}(\mathbb{R}^d)$, $\mathcal{H}_\beta(E) = \infty$.

A useful fact relating capacities and Hausdorff measures

For $\beta_1 > \beta_2 > 0$ and compact E,

$$\mathrm{Cap}_{\beta_1}(E) > 0 \Longrightarrow \mathcal{H}_{\beta_1}(E) > 0 \Longrightarrow \mathrm{Cap}_{\beta_2}(E) > 0$$

(Frostman's theorem).

Hitting probabilities for the Brownian motion

Theorem (*Kakutani, 1944*) *For a d-dim Brownian motion B*:

$$\boxed{c\,\mathrm{Cap}_{d-2}(A) \le P(B(\mathbb{R}_+) \cap A \neq \emptyset) \le \bar{c}\,\mathrm{Cap}_{d-2}(A).}$$

In particular, for $x \neq 0$,

$$P(\exists t : B(t) = x) > 0 \iff d = 1.$$

Indeed,

$$\mathrm{Cap}_\beta(\{x\}) = \begin{cases} 1, & \beta < 0, \\ 0, & \beta \ge 0. \end{cases}$$

Hitting probabilities for the Brownian motion

Theorem (*Kakutani, 1944*) *For a d-dim Brownian motion* B:

$$\boxed{c\,\mathrm{Cap}_{d-2}(A) \leq P(B(\mathbb{R}_+) \cap A \neq \emptyset) \leq \bar{c}\,\mathrm{Cap}_{d-2}(A).}$$

In particular, for $x \neq 0$,

$$P(\exists t : B(t) = x) > 0 \iff d = 1.$$

Indeed,

$$\mathrm{Cap}_\beta(\{x\}) = \begin{cases} 1, & \beta < 0, \\ 0, & \beta \geq 0. \end{cases}$$

Hitting probabilities for solutions to SPDEs

A class of SPDEs

$$Lu(t,x) = b(u(t,x)) + \underbrace{\sigma(u(t,x))\dot{W}(t,x)}_{\text{random forcing}},$$

$t > 0$, $x \in \mathbb{R}^k$.

Examples

- $L = \frac{\partial}{\partial t} - \Delta$ (heat);
- $L = \frac{\partial^2}{\partial t^2} - \Delta$ (waves);
- $L = -\Delta$ (Laplace);

with suitable initial conditions.

Digression on radomness

Analysis of physical models at different levels

- Microscopic. Systems of particles are random (quantum effects, multiple collisions, ...). Re-scale the system and pass to the limit.
- Macroscopic. Scaling corresponding to the law of large numbers. Randomness averages: PDEs or ODEs.
- Mesoscopic. Scaling corresponding to the central limit theorem. Microscopic random effects average out enough to be tractable, but do not disappear completely. The passage to the limit from microscopic to mesoscopic level leads to SPDEs or SDEs.

Rigorous formulation of the SPDE

$$u(t,x) = I_0(t,x) + \int_0^t ds \left[G(x,\cdot) * b(u(t-s,\cdot))\right](x)$$
$$+ \int_0^t \int_{\mathbb{R}^k} G(t-s,x-y)\sigma(u(s,y))W(ds,dy).$$

Back to hitting probabilities

An obvious remark

Having information on the densities of $u(t,x)$ could be useful in the study of the hitting probabilities

$$P\{u(Q) \cap A \neq \emptyset\},$$

$Q \subset \mathbb{R}_+ \times \mathbb{R}^k$.

Two levels of difficulty

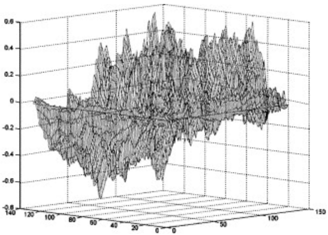
- σ constant: additive noise.
- σ non constant: multiplicative noise.

A drastic simplification

- Initial condition contribution $I_0 \equiv 0$.
- $b \equiv 0$, $\sigma \equiv 1$.

The solution to the SPDE is a Gaussian process:

$$\boxed{u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y) W(ds, dy).}$$



Some examples of fundamental solutions

For the heat equation

$$G(t,x) = (4\pi t)^{-\frac{k}{2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

For the wave equation

$$\begin{aligned}
k=1 \quad & G(t,x) = \frac{1}{2} 1_{\{|x|<t\}}, \\
k=2 \quad & G(t,x) = \frac{1}{2\pi} (t^2 - |x|^2)_+^{-\frac{1}{2}}, \\
k=3 \quad & G(t,dx) = \frac{1}{4\pi t} \sigma_t(dx).
\end{aligned}$$

Criteria for hitting probabilities

Dalang, S.-S., 2010

Sufficient conditions for the lower bound

Let $\{v(w), w \in \mathbb{R}^m\}$ be a d-dimensional random field

Assume:

1. $\forall w_1, w_2 \in \mathbb{R}^m$, $w_1 \neq w_2$, $(v(w_1), v(w_2))$ *has a density* p_{w_1,w_2}, *and there exist* $\gamma, \alpha \in]0,\infty[$ *such that*

$$p_{w_1,w_2}(z_1,z_2) \leq C \frac{1}{\|w_1 - w_2\|^{\gamma}} \exp\left(-\frac{\|z_1 - z_2\|^2}{\|w_1 - w_2\|^{\alpha}}\right),$$

$\forall z_1, z_2 \in \mathbb{R}^d$.

2. *The density* p_w *of* $v(w)$ *satisfies* $p_w(z) > 0$.

Then,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \geq c\,\mathrm{Cap}_{\frac{1}{\alpha}(\gamma - m)}(A).}$$

Sufficient conditions for the lower bound

Let $\{v(w), w \in \mathbb{R}^m\}$ be a d-dimensional random field

Assume:

1. *$\forall w_1, w_2 \in \mathbb{R}^m$, $w_1 \neq w_2$, $(v(w_1), v(w_2))$ has a density p_{w_1,w_2}, and there exist $\gamma, \alpha \in]0, \infty[$ such that*

$$p_{w_1,w_2}(z_1, z_2) \leq C \frac{1}{\|w_1 - w_2\|^{\gamma}} \exp\left(-\frac{\|z_1 - z_2\|^2}{\|w_1 - w_2\|^{\alpha}}\right),$$

   $\forall z_1, z_2 \in \mathbb{R}^d$.

2. *The density p_w of $v(w)$ satisfies $p_w(z) > 0$.*

Then,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \geq c\mathrm{Cap}_{\frac{1}{\alpha}(\gamma - m)}(A).}$$

Sufficient conditions for the lower bound

Let $\{v(w), w \in \mathbb{R}^m\}$ be a d-dimensional random field

Assume:

1. $\forall w_1, w_2 \in \mathbb{R}^m$, $w_1 \neq w_2$, $(v(w_1), v(w_2))$ *has a density* p_{w_1,w_2}, *and there exist* $\gamma, \alpha \in]0, \infty[$ *such that*

$$p_{w_1,w_2}(z_1, z_2) \leq C \frac{1}{\|w_1 - w_2\|^{\gamma}} \exp\left(-\frac{\|z_1 - z_2\|^2}{\|w_1 - w_2\|^{\alpha}}\right),$$

   $\forall z_1, z_2 \in \mathbb{R}^d$.

2. *The density* p_w *of* $v(w)$ *satisfies* $p_w(z) > 0$.

Then,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \geq c\mathrm{Cap}_{\frac{1}{\alpha}(\gamma - m)}(A).}$$

Sufficient conditions for the upper bound

$D \subset \mathbb{R}^d$ is fixed. Assume:

1. *$\forall w \in \mathbb{R}^m$, $v(w)$ has a density p_w, and* $\sup_{w \in Q, z \in D} p_w(z) \leq C$.
2. *There exists $\delta \in]0,1]$ such that $\forall q \in [1, \infty[$, $w_1, w_2 \in Q$,*

$$\|v(w_1) - v(w_2)\|_{L^q(\Omega)} \leq C\|w_1 - w_2\|^{\delta}.$$

Then, for any $\theta \in]0, d[$ and any $A \subset D$,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C\mathcal{H}_{\theta - \frac{m}{\delta}}(A).}$$

Remark: For some class of Gaussian processes,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C\mathcal{H}_{d - \frac{m}{\delta}}(A).}$$

Sufficient conditions for the upper bound

$D \subset \mathbb{R}^d$ is fixed. Assume:

1. *$\forall w \in \mathbb{R}^m$, $v(w)$ has a density p_w, and* $\sup_{w \in Q, z \in D} p_w(z) \leq C$.
2. *There exists $\delta \in]0,1]$ such that $\forall q \in [1,\infty[$, $w_1, w_2 \in Q$,*

$$\|v(w_1) - v(w_2)\|_{L^q(\Omega)} \leq C \|w_1 - w_2\|^{\delta}.$$

Then, for any $\theta \in]0, d[$ and any $A \subset D$,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C \mathcal{H}_{\theta - \frac{m}{\delta}}(A).}$$

Remark: For some class of Gaussian processes,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C \mathcal{H}_{d - \frac{m}{\delta}}(A).}$$

Sufficient conditions for the upper bound

$D \subset \mathbb{R}^d$ is fixed. Assume:

1. *$\forall w \in \mathbb{R}^m$, $v(w)$ has a density p_w, and* $\sup_{w\in Q, z\in D} p_w(z) \leq C$.
2. *There exists $\delta \in]0,1]$ such that $\forall q \in [1,\infty[$, $w_1, w_2 \in Q$,*

$$\|v(w_1) - v(w_2)\|_{L^q(\Omega)} \leq C\|w_1 - w_2\|^{\delta}.$$

Then, for any $\theta \in]0, d[$ and any $A \subset D$,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C\mathcal{H}_{\theta - \frac{m}{\delta}}(A).}$$

Remark: For some class of Gaussian processes,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C\mathcal{H}_{d - \frac{m}{\delta}}(A).}$$

Sufficient conditions for the upper bound

$D \subset \mathbb{R}^d$ is fixed. Assume:

1. $\forall w \in \mathbb{R}^m$, $v(w)$ *has a density* p_w, *and* $\sup_{w \in Q, z \in D} p_w(z) \leq C$.
2. *There exists* $\delta \in]0,1]$ *such that* $\forall q \in [1, \infty[$, $w_1, w_2 \in Q$,

$$\|v(w_1) - v(w_2)\|_{L^q(\Omega)} \leq C \|w_1 - w_2\|^\delta.$$

Then, for any $\theta \in]0, d[$ *and any* $A \subset D$,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C \mathcal{H}_{\theta - \frac{m}{\delta}}(A).}$$

Remark: For some class of Gaussian processes,

$$\boxed{P\{v(Q) \cap A \neq \emptyset\} \leq C \mathcal{H}_{d - \frac{m}{\delta}}(A).}$$

A sample of results

Stochastic heat equation

- $k = 1$, $\sigma \equiv 1$, $b \equiv 0$, space-time white noise

$$c^{-1}\mathrm{Cap}_{d-6}(A) \le P\{u(I \times J) \cap A \neq \emptyset\} \le c\mathcal{H}_{d-6}(A).$$

- $k = 1$ space-time white noise

$$c^{-1}\mathrm{Cap}_{d-6+\eta}(A) \le P\{u(I \times J) \cap A \neq \emptyset\} \le c\mathcal{H}_{d-6-\eta}(A).$$

- $k \ge 1$, noise white in time and correlated in space

(R. Dalang, D. Khoshnevishan, E. Nualart, 2007-2013)

Stochastic wave equation

- $k \geq 1$, $\sigma \equiv 1$, $b \equiv 0$, noise white in time and correlated in space (R. Dalang–S.-S, 2010)

$$c_1 \mathrm{Cap}_{d-\frac{2(k+1)}{2-\beta}}(A) \leq P\{u(I \times J) \cap A \neq \emptyset\} \leq c_2 \mathcal{H}_{d-\frac{2(k+1)}{2-\beta}}(A).$$

- $k \in \{1, 2, 3\}$, noise white in time and correlated in space (R. Dalang–S.-S, 2015)

$I \subset [0, T]$, $J \subset \mathbb{R}^k$, $A \subset [-N, N]^d$.

Stochastic Poisson equation

Fresh results!

New questions in Malliavin calculus

A difficult step in the proofs:

$$p_{w_1,w_2}(z_1, z_2) \leq C \frac{1}{\|w_1 - w_2\|^{\gamma}} \exp\left(-\frac{\|z_1 - z_2\|^2}{\|w_1 - w_2\|^{\alpha}}\right). \tag{3}$$

Comparing with Gaussian densities

$\|w_1 - w_2\|^{\gamma}$ related to the det of the cov matrix.

Remark: the Malliavin matrix $A = (\langle DX_i, DX_j\rangle_H)_{1\leq i,j\leq m}$ plays the rôle of the covariance matrix in the non-Gaussian case.

Obtaining (3) requires the study of the rate of degeneracy of the random smallest eigenvalue of A when X collapses to a constant.

Final Remarks

- Densities of Gaussian functionals can be obtained by the way of IBP formulas
- Malliavin calculus is a theory tailored to prove IBP formulas.
- The initial motivation for developing Malliavin calculus was to give a probabilistic proof of Hörmander's theorem on hypoelliptic operators. However, the theory went far beyond its initial objective.
- As usually, there is a two-way interaction: Malliavin calculus is used to approach new problems, and new problems raise new questions in the theory.

Altogether, a nice blend of ideas and techniques from different mathematical fields.

Muchas Gracias!!