Old and new spectral problems for the prolate spheroidal equation. An unified analytical approach. Application to the Connes-Moscovici spectrum which matches the zeroes of zeta

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1 Abstract

A joint work with Françoise Richard-Jung and Jean Thomann.

The main. actor in the play is the prolate spheroidal operator (of zero order) :

$$W_\Lambda = -\frac{d}{dx}(\Lambda^2 - x^2)\frac{d}{dx} + (2\pi\Lambda x)^2;$$

Λ is a parameter (bandwith parameter). It is real in a first step and after complex ; W_Λ is formally self-adjoint.

The spheroidal operators appeared in the study of the Helmoltz equation $(\Delta + k^2)\Psi = 0$ on a spheroid (prolate resp. oblate). The prolate spheroid is a rugby ball. There is a separation. of variables in prolate (resp. oblate) coordinates. The solutions of angular and radial equations are the spheroidal functions. They generalize the spherical functions (football ball).

I will recall some results in signal theory due to Slepian and all in Bell Labs 1960-1965. Signals cannot be perfectly localized in time and frequency and there is a problem in communication of signals : limited time, limited range of frequencies. This lead to the study of an integral convolution operator Q_Λ with a sine cardinal kernel. Its first eigenvalues are very near of 1 and afterwards they fall abruptly and became very small. Then the study of eigenfunctions become difficult. Slepian and all made a remarkable discovery : the operator Q_Λ commutes with the prolate differential operator W_λ. Then the eigenfunctions are the prolate spheroidal functions and it is possible to use a singular version of the Sturm-Liouville theory on $[-\Lambda, \Lambda]$ for their study. Slepian called this ‘the lucky accident” and said that it remains a mystery. I will try to explain the origin of the miracle.

Around 2007, Alain Connes and Matilde Marcolli guessed some (complicated) relations between the above picture and the zeroes of the Riemann zeta function.

Before, in 1998, A. Connes discovered a self-adjoint extension of W_Λ to the real line.

Recently A. Connes and H. Moscovici studied the (even) spectrum of W_Λ on $[\Lambda, +\infty]$. They made a remarkable discovery : the spectrum is discrete, it contains a replica of the classical (even) spectrum (on $[-\Lambda, \Lambda]$) but also a new part (negative eigenvalues) and if $\Lambda = \sqrt{2}$, then this part matches in the ultraviolet the squares of the imaginary parts of the non trivial zeros of the zeta function.

I will present a new theory of spectra of rational linear second order operators elaborated with Françoise and Jean. We define analytic spectra.

We skip the reality conditions and the SL conditions. We choose singular points as boundaries and we define boundary conditions using the structure of spaces of singular solutions (and symmetry if any). In the regular-singular case, a line of boundary condition is given by an eigenline of the monodromy (the idea appeared already in the fundamental article of Schrödinger on the hydrogen atom). In the irregular case it is more delicate and summability is needed.

I will explain how to extend the study of the CM spectrum using our new "geometric" approach in place of L^2 technics. I denote μ the spectral parameter.

The new approach allows :

1. The "construction" (by analytic matching) of a 'functional determinant" : a function $F(\Lambda, \mu)$ on $\mathbb{R}^* \times \mathbb{C}$ such that, for Λ fixed, the zeros are the eigenvalues. This function is real analytic in Λ and entire in μ of exponential growth $\leq 1/2$. There are (partially conjectural) product formulae.
2. It is possible to define an extension of the CM spectrum for $\Lambda \in \mathbb{C}^*$, more precisely it is necessary to use $\log \Lambda$. Then we can consider the analytic extensions of the CM eigenvalues. For the classical eigenvalues this was done by Meixner and Schäfke in their fundamental monography. We conjecture that the eigenvalues are related by analytic continuation as in the quartic oscillator case (Bender-Wu).
3. We get accurate methods of computation of the eigenvalues (using summation and analytic continuation with SageMath).
4. It is possible to interpret the CM eigenvalues by pullback by a Riemann-Hilbert map of an algebraic curve (explicitely computed) on the character variety of the prolate equations, an affine cubic surface (explicitely computed). The classical spectrum is the pullback of a (double) line on the cubic surface.